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A NOTE ON THE KAZDAN-WARNER TYPE CONDITIONS

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Abstract

We consider prescribing Gaussian curvature on a 2-sphere S^2. There are well-known Kazdan-Warner and Bourguinon-Ezin necessary conditions for a function K to be the Gaussian curvature of some pointwise conformal metric. Then are those necessary conditions also sufficient? This is a problem of common concern and has been left open for a few years. In this paper, we answer the question negatively. First, we show that if K is rotationally symmetric and is monotone in the region where $K > 0$, then the problem has no rationally symmetric solution. Then we provide a family of functions K satisfying the Kazdan-Warner and Bourguinon-Ezin conditions, for which the problem has no solution at all. We also consider prescribing scalar curvature on S^n for $n \geq 3$. We prove the nonexistence of rationally symmetric solution for the above-mentioned functions.

1. Introduction

Given a continuous function $K(x)$ on a compact surface S, it is interesting to know that whether it can be the Gaussian curvature of some metric. In practice, one often seeks the unknown metric by picking a basic metric g_0, and then pointwise conformally deforms it to the desired metric g. If we let $g = e^{2u} g_0$, then it is equivalent to solving the following nonlinear elliptic equation:

$$-\Delta_0 u + K_0(x) = K(x) e^{2u(x)}, \qquad x \in S, \tag{1.1}$$

where Δ_0 and $K_0(x)$ are the Laplacian and the Gaussian curvature of g_0. In the last few years, a lot of work has been done to understand this (cf. [2], $\cdots$, [8], [11], $\cdots$, [19] and the references therein).

For equation (1.1) to have a solution, the function $K(x)$ must satisfy the obvious Gauss-Bonnet sign condition

$$\int_S K(x) e^{2u}\, dx = 2\pi\chi, \tag{1.2}$$

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where χ is the Euler characteristics of the surface S.

On the standard sphere S^2, equation (1.1) reads as

$$(*) \qquad -\Delta_0 u + 1 = K(x)e^{2u}, \qquad x \in S^2.$$

Now, besides (1.2), there are other well-known obstructions, the Kazdan-Warner conditions [16]

$$(1.3) \qquad \int_{S^2} \nabla K \cdot \nabla \phi_i e^{2u}\, dA_0 = 0, \qquad i = 1, 2, 3,$$

where ϕ_i are the first spherical harmonic functions.

These conditions give rise to many examples of $K(x)$ for which $(*)$ has no solution. In particular, a monotone rotationally symmetric function K admits no solution.

Later, Bourguignon and Ezin [1] generalized the condition to

$$(1.4) \qquad \int_{S^2} X(K)e^{2u}\, dA_0 = 0,$$

where X is any conformal vector field on S^2.

On the other hand, many authors have found various sufficient conditions for $(*)$ to have a solution. However, there is still an obvious gap between the necessary ones and the sufficient ones. Then what are the necessary and sufficient conditions? In virtue of (1.2), (1.4), one may probably guess the following condition:

(1.5) $K > 0$ somewhere and there exists u such that (1.4) holds

would be such a candidate.

Let us first consider the case where K is rotationally symmetric. Conditions (1.5) become

(1.6) $K > 0$ somewhere and ∇K changes signs.

Now,

(1.7) is (1.6) a sufficient condition for $(*)$ to have a solution?

This has been an open problem for many years (cf. [15]).

In their recent paper, Xu and Yang [19] presented a family of rotationally symmetric function K_ε satisfying (1.6) and having a monotone limit. They showed that for ε small, $(*)$ cannot have a rotationally symmetric solution with $K = K_\varepsilon$, by proving a compactness of such a family of solutions u_ε. This result is interesting. It shed some doubt on the sufficiency of the condition (1.6). However, one does not know if there are any nonrotationally-symmetric solutions for their functions K_ε, hence it is still not clear if (1.6) can be sufficient or not.

In this paper, we give the question (1.7) a negative answer. We provide a family of rotationally symmetric functions K satisfying (1.6) for which equation $(*)$ has no solution at all.

First we show the nonexistence of rotationally symmetric solution.

Theorem 1. *Let K be rotationally symmetric. If*

$$K \text{ is monotone in the region where } K > 0, \tag{1.8}$$

then problem $(*)$ has no rotationally symmetric solution.

This generalizes Xu and Yang's result, since their family of functions K_ε satisfies (1.8).

Although we believe that for all such functions K, there is no solution at all, we are not able to prove it by now. However, we can show this for a family of such functions.

Theorem 2. *There exists a family of functions K satisfying the Kazdan-Warner type conditions* (1.6), *for which the problem $(*)$ has no solution.*

This theorem gives (1.7) a negative answer. (Please see §3 for the details of such family of functions.)

In [19], Xu and Yang also proved

Proposition. *Let K be rotationally symmetric. Assume that*

(i) *K is nondegenerate;*

(ii)

$$\nabla K \text{ changes signs in the region where } K > 0. \tag{1.9}$$

Then problem $(*)$ has a solution.

The above results and many other existence results tend to lead people to believe that what really count is whether ∇K changes signs in the region where $K > 0$. And we guess that for rotationally symmetric K, condition (1.9), instead of (1.6), would be the necessary and sufficient condition for $(*)$ to have a solution. In order to confirm this guess, one needs to work on two aspects:

(i) Drop the nondegeneracy assumption on K in Xu and Yang's existence result.

(ii) Improve our Theorems 1 or 2. Show that for functions satisfying (1.8), there is no solution at all.

We also consider prescribing scalar curvature problems in higher dimensions. On S^n, $n \geq 3$, one would like to know what kinds of functions $R(x)$ can be realized as the scalar curvature of some pointwise conformal metric. This is equivalent to solving the following equation:

$$(**)\qquad -\Delta_0 u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} R(x) u^{(n+2)/(n-2)}, \qquad x \in S^n,$$

where Δ_0 is the Laplacian of the standard S^n.

There are similar obstructions for the solution of $(**)$ found by Kazdan and Warner [16] and then generalized by Bourguignon and Ezin [1]. In the case where R is rotationally symmetric, the obstructions read as

$$R > 0 \text{ somewhere and } \nabla R \text{ changes signs}.$$

Again one would like to ask: Is this a sufficient condition? We incline to a negative answer. We can prove the nonexistence of rotationally symmetric solutions.

Theorem 3. *Let R be rotationally symmetric. If R is monotone in the region where $R > 0$, then equation $(**)$ has no rotationally symmetric solution.*

At this stage, we are not able to prove that for some of these kinds of functions R, there is no solution at all; however we guess one should be able to do so.

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2. Proof for nonexistence of radial solutions

In this section, we prove

Theorem 1. *Let K be rotationally symmetric. If K is monotone in the region where $K > 0$, then problem $(*)$ has no rotationally symmetric solution.*

Proof. For simplicity, we make a stereographic projection from S^2 to a Euclidean plane R^2, then $(*)$ is equivalent to the following equation:

$$(\overline{*})\qquad -\Delta u = R(x) e^u, \qquad x \in R^2,$$

where $R(x) = 2K(x)$, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, and u is another unknown function satisfying $\lim_{|x|\to\infty}(u(x)/\ln|x|) = -4$.

We assume that R is bounded, continuous and radially symmetric, $R = R(r)$ with $r = |x|$. Without loss of generality, we may also assume that there is $r_0 > 0$, such that

(i) $R(r) \le 0$ for $r \le r_0$,

(ii) $R(r) > 0$ and $R'(r) \ge 0$, $R'(r) \not\equiv 0$ for $r > r_0$.

We are going to show that equation $(\overline{*})$ has no radial solution.

Assume to the contrary, there is a function $u = u(r)$ that solves $(\overline{*})$. Then multiply the both sides of $(\overline{*})$ by $x\cdot\nabla u$ and integrate over the region $\Omega_\lambda \equiv B_\lambda(0)\backslash B_{r_0}(0)$ for $\lambda > r_0$. By a straightforward calculation, one can see that the left-hand side becomes

$$\int_{\partial\Omega_\lambda}\left(\frac{1}{2}x\cdot\nu|\nabla u|^2 - \frac{\partial u}{\partial\nu}x\cdot\nabla u\right)ds, \tag{2.1}$$

and the right-hand side becomes

$$-2\int_{\Omega_\lambda} Re^u\,dx - \int_{\Omega_\lambda} x\cdot\nabla Re^u\,dx + \int_{\partial\Omega_\lambda} x\cdot\nu Re^u\,ds, \tag{2.2}$$

where ν is the unit outward normal vector of $\partial\Omega_\lambda$.

Now let $\lambda\to\infty$. By the asymptotic behavior of u at infinity $u\sim -4\ln r$, we have

$$\int_{\partial B_\lambda}\left(\frac{1}{2}x\cdot\nu|\nabla u|^2 - \frac{\partial u}{\partial\nu}x\cdot\nabla u\right)ds \to -16\pi.$$

Consequently, (2.1) is reduced to

$$\pi r_0^2(u'(r_0))^2 - 16\pi. \tag{2.3}$$

While on the other hand, applying the Gauss-Bonnet formula $\int_{R^2} Re^u\,dx = 8\pi$, and the asymptotic behavior of u at infinity $u\sim -4\ln r$, and noting that $R = 0$ on ∂B_{r_0}, we see that (2.2) takes the form

$$-16\pi + 2\int_{B_{r_0}(0)} Re^u\,dx - \int_{R^2\backslash B_{r_0}} x\cdot\nabla Re^u\,dx. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\pi r_0^2(u'(r_0))^2 = 2\int_{B_{r_0}(0)} Re^u\,dx - \int_{R^2\backslash B_{r_0}} x\cdot\nabla Re^u\,dx,$$

which is impossible, since the left-hand side is nonnegative while the right-hand side is negative. Therefore, equation $(\overline{*})$ cannot have a radial solution.

3. Symmetry of solutions

In this section, we prove Theorem 2. As we did in §2, for simplicity, we make a stereographic projection and consider the equation

$$-\Delta u = R(x)e^u, \qquad x\in R^2. \tag{$\overline{*}$}$$

We are going to find a family of functions R satisfying both Kazdan-Warner and Bouguignon-Ezin conditions, for which equation $(\overline{*})$ has no solution. Based on our result in §2, what we need to show here is that for such a family of functions, all the solutions of $(\overline{*})$ are radially symmetric. In order to apply our Theorem 1, we still assume that R is bounded, continuous, and, that

(3.1) $R = R(r)$ and R is monotone in the region where R is positive.

We shall use the method of moving planes to prove

Theorem 3.1. *Assume that R satisfies*

$$R'(r) + 2rR(r) \leq 0. \tag{3.2}$$

Then all the solutions of $(\overline{*})$ are radially symmetric about the origin.

Remark. A candidate for such function R satisfying the Kazdan-Warner type condition (1.6) and (3.1) and (3.2) would look like the following: It is positive and monotone decreasing for $r < r_0$, it is negative and its derivative changes signs in the region where $r > r_0$, and it satisfies (3.2). The following is one of the examples of such functions:

$$R(r) = \begin{cases} e^{-r^2} - 1/e & \text{for } r \leq a, \\ e^{-r^2} - a/er & \text{for } r > a, \end{cases}$$

where $r_0 = 1$, and a is a sufficiently large number.

In order to apply the method of moving planes to equation $(\overline{*})$, it requires the function R be monotone decreasing. However, we want our R to satisfy conditions of Kazdan-Warner type; hence it must not be monotone in the whole domain.

To circumvent this difficulty, we introduce a new unknown function $v(x) \equiv u(x) - |x|^2$. Obviously, v satisfies the following equation:

$$-\Delta v - 4 = \tilde{R}e^{v}. \tag{3.3}$$

Now our new function $\tilde{R}(r) = R(r)e^{r^2}$ is monotone decreasing due to assumption (3.2). Then we can apply the method of moving planes similar to our previous paper [10] to prove that v must be radially symmetric about the origin. Therefore, u must also be radially symmetric. For completeness, we still present our proof in the following.

To prove the symmetry, we move a family of lines which are orthogonal to a given direction from negative infinity to a critical position, and then show that the solution is symmetric in that direction about the critical position. We also show that the solution is strictly increasing before the critical position. Since the direction can be chosen arbitrarily, we conclude that the solution must be radially symmetric.

Assume that $v(x)$ is a solution of (3.3). Without loss of generality, we show the monotonicity and symmetry of the solution in the x_1 direction.

For $\lambda \in R^1$, let $\Sigma_\lambda = \{(x_1, x_2) | x_1 < \lambda\}$ and $T_\lambda = \partial\Sigma_\lambda = \{(x_1, x_2) | x_1 = \lambda\}$. Let $x^\lambda = (2\lambda - x_1, x_2)$ be the reflection point of $x = (x_1, x_2)$ about the line T_λ.

Define $w_\lambda(x) = v(x^\lambda) - v(x)$ and $\overline{w}_\lambda(x) = w_\lambda(x)/g(x)$ with $g(x) = \ln(-x_1 + 2) + \ln(1 + x_1^2 + x_2^2)$. In the following we assume that $\lambda \leq 0$ and $x \in \Sigma_\lambda$. Obviously w_λ and $\overline{w}_\lambda$ are well defined and $g(x) > 0$. A straightforward calculation yields that

$$\text{(3.4)} \quad \Delta w_\lambda(x) + \widetilde{R}(x)(\exp \psi(x)) w_\lambda(x) = (\widetilde{R}(x) - \widetilde{R}(x^\lambda)) \exp v(x^\lambda) \leq 0,$$

where $\psi(x)$ is some real number between $v(x)$ and $v(x^\lambda)$. Consequently

$$\text{(3.5)} \qquad \Delta \overline{w}_\lambda + \frac{1}{g} \nabla g \cdot \nabla \overline{w}_\lambda + \left(\widetilde{R}(x) \exp \psi + \frac{\Delta g}{g} \right) \overline{w}_\lambda \leq 0.$$

We are going to show that $\overline{w}_0(x) \equiv 0$. To this end, we need the following simple version of the

Maximum Principle and the Hopf Lemma. *Assume that v satisfies $\Delta v + b_i v_i + cv \leq 0$ and $v \geq 0$ in a domain Ω of R^2 with smooth boundary $\partial\Omega$ and $v = 0$ on $\partial\Omega$.*

1. *If v vanishes at some point in Ω, then $v \equiv 0$ in Ω.*

2. *If $v \not\equiv 0$ in Ω, then on $\partial\Omega$, the exterior normal derivative $\partial v / \partial n < 0$.*

We also need

Lemma 3.2. (i) *For each fixed λ, $\overline{w}_\lambda(x) \to 0$, as $|x| \to \infty$.*

(ii) *There exists $R_0 > 0$ independent of λ, such that if x^0 is a minimum point of $\overline{w}_\lambda(x)$ and $\overline{w}_\lambda(x^0) < 0$, then $|x^0| < R_0$.*

Proof. (i) One can easily see that as $|x| \to \infty$, $g(x) \to +\infty$. While on the other hand, by the asymptotic behavior of v, $w_\lambda(x)$ is bounded. Hence $\overline{w}_\lambda(x) \to 0$ as $|x| \to \infty$.

(ii) To prove this part we first note that

$$\frac{\Delta g}{g} = \frac{1}{\ln(-x_1 + 2) + \ln(1 + x_1^2 + x_2^2)} \left[\frac{-1}{(x_1 - 2)^2} + \frac{4}{(1 + x_1^2 + x_2^2)^2} \right]$$

and $\psi(x^0) \leq \max\{v(x^0), v(x^{0^\lambda})\}e = v(x^0)$. Then the asymptotic behavior of v implies that there exists some number R_0, such that

$$\widetilde{R}(x^0) \exp \psi(x^0) + \Delta g / g < 0 \quad \text{if } |x^0| > R_0.$$

Now conclusion (ii) of Lemma 3.2 follows directly from equation (3.5).

Proof of Theorem 3.1.

Step 1. Let R_0 be given in Lemma 3.2. We will show that if $\lambda < -R_0$, then $\overline{w}_\lambda(x) \geq 0$ for $x \in \Sigma_\lambda$.

In fact, suppose to the contrary, there is some $x \in \Sigma_\lambda$, such that $\overline{w}_\lambda(x) < 0$. Then by (i) of Lemma 3.2, $\lim_{|x|\to+\infty} \overline{w}_\lambda(x) = 0$; hence one can find a point $x^0 \in \Sigma_\lambda$ such that $\overline{w}_\lambda(x^0) = \min_{\Sigma_\lambda} \overline{w}_\lambda < 0$. This contradicts (ii) of Lemma 3.2.

Step 2. Let λ_0 be the largest possible value of λ such that $\overline{w}_\lambda(x) \geq 0$ for $x \in \Sigma_\lambda$ and $\lambda \leq \lambda_0$. Then $\lambda_0 \geq 0$. We show this by contradiction. Suppose $\lambda_0 < 0$. Then by (3.4) and the fact that $\widetilde{R}$ is not a constant (since R is not monotone), applying the maximum principle and the Hopf Lemma, we have $w_{\lambda_0} > 0$ and hence $\overline{w}_{\lambda_0} > 0$ in Σ_{λ_0} and $\partial \overline{w}_{\lambda_0}/\partial x_1 < 0$ on T_{λ_0}.

On the other hand, by the definition of λ_0, there exists a sequence of real numbers $\lambda_k \searrow \lambda_0$ such that $\overline{w}_{\lambda_k}(x) < 0$ for some $x \in \Sigma_{\lambda_k}$. Let x^k be a minimum point of $\overline{w}_{\lambda_k}$. Then $\overline{w}_{\lambda_k}(x^k) < 0$ and $\nabla \overline{w}_{\lambda_k}(x^k) = 0$ for $k = 1, 2, 3, \ldots$, so Lemma 3.2 implies that $|x^k| \leq R_0$; hence there is a subsequence of $\{x^k\}$ converging to some point $x^0 \in R^2$. Obviously $x^0 \in \Sigma_{\lambda_0} \cup T_{\lambda_0}$, $\overline{w}_{\lambda_0}(x^0) \leq 0$ and $\nabla \overline{\omega}_{\lambda_0}(x^0) = 0$. This is impossible.

Step 3. By moving the family of lines T_λ from negative infinity to the origin we conclude that $w_0(x) \geq 0$, or, in other words, $v(-x_1, x_2) \leq v(x_1, x_2)$ for $x_1 > 0$. Then using an entirely similar approach, moving the family of lines T_λ from positive infinity to the origin, we can show that $v(-x_1, x_2) \geq v(x_1, x_2)$. Therefore $v(-x_1, x_2) \equiv v(x_1, x_2)$, which completes the proof of the theorem.

4. Higher dimensional case

In this section, we prove

Theorem 3. *Let R be rotationally symmetric. If R is monotone in the region where $R > 0$, then problem $(**)$ has no rotationally symmetric solution.*

Proof. As in §2, for simplicity, we make a stereographic projection from S^n to R^n. Then $(**)$ is equivalent to the following equation:

$$(\overline{**}) \qquad -\Delta u = R(x)u^p, \qquad x \in R^n,$$

where Δ is Euclidean Laplacian, $p = (n+2)/(n-2)$, $u > 0$ and satisfies $u \sim |x|^{2-n}$ near infinity.

We assume that R is bounded, continuous, and radially symmetric, $R = R(r)$ with $r = |x|$. Without loss of generality, we may also assume that there is $r_0 > 0$, such that

(i) $R(r) \le 0$ for $r \le r_0$,

(ii) $R(r) > 0$ and $R'(r) \ge 0$, $R'(r) \not\equiv 0$ for $r > r_0$.

We are going to show that equation $(\overline{**})$ has no radical solution.

Assume to the contrary that there is a function $u = u(r)$ which solves $(\overline{**})$. Then multiply the both sides of $(\overline{**})$ by $x \cdot \nabla u$ and integrate over the region $\Omega_\lambda \equiv B_\lambda(0) \backslash B_{r_0}(0)$ for $\lambda > r_0$. By a straightforward calculation, one can see that the left-hand side becomes

$$\int_{\partial\Omega_\lambda} \left(\frac{1}{2} x \cdot \nu |\nabla u|^2 - \frac{\partial u}{\partial \nu} x \cdot \nabla u\right) dS + \left(1 - \frac{n}{2}\right) \int_{\Omega_\lambda} |\nabla u|^2 \, dx \tag{4.1}$$

and the right-hand side becomes

$$\begin{aligned} &\left(1 - \frac{n}{2}\right) \int_{\Omega_\lambda} R u^{p+1} \, dx - \frac{n-2}{2n} \int_{\Omega_\lambda} x \cdot \nabla R u^{p+1} \, dx \\ &\quad + \frac{n-2}{2n} \int_{\partial\Omega_\lambda} x \cdot \nu R u^{p+1} \, dS. \end{aligned} \tag{4.2}$$

In order to simplify our result, this time by multiplying the both sides of $(\overline{**})$ by u and integrating over Ω_λ we get

$$\int_{\Omega_\lambda} |\nabla u|^2 \, dx - \int_{\partial\Omega_\lambda} u \frac{\partial u}{\partial \nu} \, dS = \int_{\Omega_\lambda} R u^{p+1} \, dx. \tag{4.3}$$

Now let $\lambda \to \infty$. By the asymptotic behavior $u \sim r^{2-n}$, the fact $R(r_0) = 0$ and (4.1), (4.2), (4.3) we arrive at

$$\begin{aligned} &\int_{\partial B_{r_0}(0)} \left[\frac{r_0}{2} (u'(r))^2 + \left(\frac{n}{2} - 1\right) u u'(r)\right] dS \\ &\qquad = -\frac{n-2}{2n} \int_{R^2 \backslash B_{r_0}(0)} r R'(r) u^{p+1} \, dx. \end{aligned} \tag{4.4}$$

Finally to find a contradiction, we apply the maximum principle to the region $B_{r_0}(0)$. By our assumption on R, we have

$$\Delta u = -R u^p \ge 0,$$

which implies that

$$u(r_0) = \max_{B_{r_0}(0)} u,$$

so that $u'(r_0) \geq 0$, an obvious contradiction to (4.4). Hence the proof is complete.

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